

The effect of range and bandwidth on the round complexity in the congested clique model^{*}

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Abstract. The congested clique model is a message-passing model of distributed computation where k players communicate with each other over a complete network. Here we consider synchronous protocols in which communication happens in rounds (we allow them to be randomized with public coins). In the *unicast* communication mode, each player i has her own n -bit input x_i and may send $k - 1$ different b -bit messages through each of her $k - 1$ communication links in each round. On the other end is the *broadcast* communication mode, where each player can only broadcast a single message over all her links in each round. The goal of this paper is to complete our Brief Announcement at PODC 2015, where we initiated the study of the space that lies between the two extremes. For that purpose, we parametrize the congested clique model by two values: the *range* r , which is the maximum number of different messages a player is allowed to send in each round, and the *bandwidth* b , which is the maximum size of these messages. We show that the space between the unicast and broadcast congested clique models is very rich and interesting. For instance, we show that the round complexity of the pairwise set-disjointness function PWDISJ is completely sensitive to the range r . This translates into a $\Omega(k)$ gap between the unicast ($r = k - 1$) and the broadcast ($r = 1$) modes. Moreover, provided that $r \geq 2$ and $rb/\log r = O(k)$, the round complexity of PWDISJ is $\Theta(n/k \log r)$. On the other hand, we also prove that the behavior of PWDISJ is exceptional: almost every boolean function f has maximal round complexity $\Theta(n/b)$. Finally, we prove that $\min\left(\left\lceil \frac{b'}{\lfloor \log r \rfloor} \right\rceil, \left\lceil \frac{r'}{r-1} \right\rceil \left\lceil \frac{b'}{b} \right\rceil\right)$ is an upper bound for the gap between the round complexities with parameters (b, r) and parameters (b', r') of any boolean function.

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1 Introduction

In this paper we study a synchronous, message-passing model of distributed computation where the underlying communication network is a complete graph. Therefore, the only obstacle to perform any task is due to *congestion*. In fact, the main theoretical purpose of this model, known as *congested clique*, is to serve as a basic model for understanding the role played by congestion in distributed computation [14, 15, 21, 25, 27, 28]. (Besides this, there are interesting connections between the congested clique model and popular systems such as MapReduce [20].)

The model is defined as follows. There are k players. Each player has her own n -bit input x_i and they all collaborate in order to compute a joint boolean function $f(x_1, \dots, x_k)$. They communicate with each other in synchronous rounds. More precisely, each of the k players may send up to $k - 1$ different b -bit messages through each of her $k - 1$ communication links. A protocol that computes f stops when every player knows the output. We use the number of rounds as the goodness metric to be minimized. The absolute minimum of this parameter is what we call *round complexity*. In this paper all protocols are allowed to be randomized with public coins. More precisely, the k players have access to a common infinite string of independent random bits. Protocols may return the wrong answer with probability at most ϵ , for some fixed, small $\epsilon > 0$.

Most work on this (unicast) congested clique model considers the joint input as a graph G by giving to each player i the boolean vector $x_i \in \{0, 1\}^n$, which is the indicator function of her neighborhood in G . Note that in this case $n = k$ and, therefore, the total number of bits exchanged in each round is bn^2 . Unfortunately, due to the huge number of bits transmitted globally per round (even for $b = 1$), no lower bound is known for this model. Drucker, Kuhn and Oshman gave in [15] an explanation for this difficulty. They proved that in this model it is possible to simulate powerful classes of bounded-depth circuits (and therefore lower bounds in the congested clique would yield lower bounds in circuit complexity). The intrinsic power of the (synchronous) congested clique model has allowed some authors [10, 14, 19, 21] to provide extremely fast protocols for some natural graph problems (assuming always that $b = \log n$, following the spirit of the *CONGEST* model [29]).

In the broadcast version of the congested clique model, each player can only broadcast a single b -bit message over all her links in each round [15]. This setting is equivalent to the multi-party, number-in-hand computation model, where communication takes place in a shared blackboard [1, 2, 5–7, 15]. In fact, writing a message \mathcal{M} on the blackboard is equivalent to broadcasting \mathcal{M} . In this setting, the number of transmitted bits per round decreases from bn^2 to bn . Therefore, obtaining lower bounds using communication complexity reductions becomes possible. For instance, detecting deterministically a triangle in the input graph G requires $\Omega(n/(e^{\mathcal{O}(\sqrt{\log n})}b))$ rounds [15]. On the other hand, fast protocols are also known in the broadcast congested clique model [1, 2, 18, 23].

There is a particular boolean function that we are going to use throughout this paper. This function, that we call *pairwise set-disjointness*, is defined below.

Definition 1. Let $k = 2k'$. Let $x = (x_1 \dots x_k) \in (\{0, 1\}^n)^k$. Each x_i is the indicator vector of a subset $X_i \subseteq \{1, \dots, n\}$. Function pairwise set-disjointness PWDISJ is defined by: $\text{PWDISJ}(x) = 1$ if $\forall 1 \leq i \leq k', X_i \cap X_{i+k'} = \emptyset$; and $\text{PWDISJ}(x) = 0$ otherwise.

Our goal is to complete the work of [4], where we initiated the study of the round complexity of boolean functions according to two parameters of the model:

- The *range* r : the maximum number of different messages a player can send over her links in one round.
- The *bandwidth* b : the maximum size, in bits, of each of these messages.

By analogy with the notation introduced in [15], we denote this model by $\text{CLIQUE-RCAST}_{r \times b}$. Note that the two extreme cases $r = 1$ and $r = k - 1$, which correspond to the broadcast and the unicast communication modes, are the cases already considered in the literature. More precisely,

$$\begin{aligned} \text{CLIQUE-RCAST}_{(k-1) \times b} &= \text{CLIQUE-UCAST}_b, \\ \text{CLIQUE-RCAST}_{1 \times b} &= \text{CLIQUE-BCAST}_b. \end{aligned}$$

Note also that, if the available bandwidth b is too small, then having a big range r becomes useless, since the number of possible different messages with a bandwidth b is 2^b . More precisely,

$$\forall r \geq 2^b, \text{CLIQUE-RCAST}_{r \times b} = \text{CLIQUE-RCAST}_{2^b \times b} = \text{CLIQUE-UCAST}_b.$$

Thus, in the sequel, we will assume that $r \leq 2^b$. We denote by $\text{ROUND}_{r \times b}(f)$ the round complexity of function f . That is, $\text{ROUND}_{r \times b}(f)$ denotes the minimal number of rounds needed by any k -player protocol in $\text{CLIQUE-RCAST}_{r \times b}$ for computing f . We also denote,

$$\begin{aligned} \mathbf{UROUND}_b(f) &= \text{ROUND}_{(k-1) \times b}(f), \\ \mathbf{BROUND}_b(f) &= \text{ROUND}_{1 \times b}(f). \end{aligned}$$

A protocol in $\text{CLIQUE-RCAST}_{r \times b}$ is said to be a *broadcasting protocol* if it consists of every player broadcasting its complete input. Obviously, for any function f , there exists a broadcasting protocol which computes f , and we get the universal bound $\text{ROUND}_{r \times b}(f) \leq \mathbf{BROUND}_b(f) \leq \lceil n/b \rceil$. In order to understand the role played by the range r and the bandwidth b in the round complexity of the congested clique model we define the following ratio.

$$I_{r' \times b'}^{r \times b}(f) = \frac{\text{ROUND}_{r \times b}(f)}{\text{ROUND}_{r' \times b'}(f)}.$$

The values above obviously depend on k , n and ϵ . But we omit them in order to avoid heavy notation. Finally, by taking the uniform probability over $\{0, 1\}^{\{0, 1\}^{kn}}$, we also consider what happens with random boolean functions. For instance, we compute probabilities such as $\Pr\{I_{r' \times b'}^{r \times b}(f) = \alpha\}$, for fixed α .

1.1 Our results

In Section 2 we compare the broadcast model and the unicast model. For that purpose we consider the pairwise set-disjointness function PWDISJ. We prove that $\mathbf{U}ROUND_b(\text{PWDISJ}) = \mathcal{O}(n/kb)$ while $\mathbf{B}ROUND_b(\text{PWDISJ}) = \Omega(n/b)$. In other words, $\Gamma_{(k-1) \times b}^{1 \times b}(\text{PWDISJ}) = \Omega(k)$. This gives a large gap between the unicast and broadcast congested clique models, that grows at least linearly with k .

In Section 3 we prove that the round complexity of PWDISJ is completely sensitive to the range r even in the intermediate values between unicast and broadcast. More precisely, we prove that for k sufficiently large and for $r \geq 2$ such that $rb/\log r = \mathcal{O}(k)$ the following holds: $\text{ROUND}_{r \times b}(\text{PWDISJ}) = \Theta(n/k \log r)$. Then, we give some interpretations to this result. In particular, we conclude that $\Gamma_{r' \times \log k}^{r \times \log k}(\text{PWDISJ}) = \Theta(\log r'/\log r)$ for every $r' \geq r \geq 2$. Note that the logarithmic bandwidth is the most studied case in the congested clique model, and this result yields a hierarchy of models of different computational power according to the range r for this case.

In Section 4 we prove that almost every boolean function f satisfies that $\mathbf{U}ROUND_b(f) = \mathbf{B}ROUND_b(f) = \lceil n/b \rceil$, provided that k is sufficiently large and that $0 \leq \epsilon \leq 0.2$. In other words, $\Gamma_{(k-1) \times b}^{1 \times b}(f) = 1$ for almost every f . This means that the gap we found in Section 2 for function PWDISJ is exceptional and that the power given by having $r > 1$ is almost always useless. Nevertheless, as pointed out by Drucker et al. [15], finding for $k = n$ an explicit boolean function f with the behavior $\mathbf{U}ROUND_b(f) = \omega(1)$ is (equivalent to solving) a long-standing open problem in circuit complexity theory.

The goal of Section 5 is to compare models with different combinations of range and bandwidth for arbitrary boolean functions f . For doing this we analyze the ratio $\Gamma_{r' \times b'}^{r \times b}(f)$. We make the following observation: for almost every function f we have $\Gamma_{r' \times b'}^{r \times b}(f) = \Theta(b'/b)$. Moreover, if $r \geq r'$ or $r = 2^b$ then $\Gamma_{r' \times b'}^{r \times b}(f) \leq \lceil b'/b \rceil$ for every boolean function f . The general upper bound we obtain is the following $\Gamma_{r' \times b'}^{r \times b}(f) \leq \min \left(\left\lceil \frac{b'}{\lceil \log r \rceil} \right\rceil, \left\lceil \frac{r'}{r-1} \right\rceil \left\lceil \frac{b'}{b} \right\rceil \right)$, for $r \geq 2$.

1.2 Related work: The asynchronous case

The congested clique model with bandwidth $b = 1$ –that is, the multiplayer, number-in-hand, message passing model– was introduced by Dolev and Feder [13]. The main difference with our setting is that the original model was *asynchronous*. Hence, protocols, instead of being designed to minimize the number of rounds, were designed to minimize the number of exchanged bits. The first communication complexity lower bounds were obtained by Duris and Rolim [16].

Recently, new techniques and new results have been developed, and tight bounds for the communication complexity of different functions have been obtained. In [30] the authors introduced the symmetrization technique and were able to prove tight $\Omega(nk)$ lower bounds for several direct-sum-like functions such as coordinate-wise AND or coordinate-wise OR. These lower bounds also apply

in the blackboard communication mode, where players write messages on a blackboard, visible to everybody. (Note that, in the asynchronous setting, the communication complexity in the blackboard mode gives stronger lower bounds than the communication complexity in the message-passing, point-to-point mode.) This symmetrization technique has been used and developed by other authors as well [26, 31].

It is important to point out that there exists a strict separation between the blackboard communication mode and the message-passing communication mode. For instance, the communication complexity for computing the multiparty set-disjointness function is $\Theta(n \log k + k)$ in the blackboard communication mode [9] and it is $\Theta(nk)$ in the message-passing communication mode [8]. These results on set-disjointness were obtained by using information complexity, a theory introduced in [11]. Information complexity turned out to be an extremely useful theory for proving communication complexity lower bounds [3, 12, 17].

2 A gap in the round complexity of broadcast versus unicast

The first question we would like to answer is the following: How much do we gain if, instead of broadcasting, we have the possibility of sending at least two different messages in each round? This seems to be a simple question. But it is a fundamental one if we want to understand the role played by the range in the congested clique model. For answering this we use the pairwise set-disjointness function PWDISJ defined in Section 1.

Theorem 1. $\mathbf{URound}_b(\text{PWDISJ}) = \mathcal{O}(n/kb)$.

Proof. We prove that $\mathbf{URound}_b(\text{PWDISJ}) \leq \left\lceil \frac{\lceil n/k \rceil}{b} \right\rceil + 1$. The protocol is as follows. Let $T = \left\lceil \frac{\lceil n/k \rceil}{b} \right\rceil$. For every $1 \leq t \leq T$, let

$$w_{i,j,t} = (x_i)_{(j-1)\lceil n/k \rceil + (t-1)b + 1}, \dots, (x_i)_{(j-1)\lceil n/k \rceil + tb}.$$

Round $1 \leq t \leq T$. Each player i sends to each player j (including itself) the b bits of $w_{i,j,t}$.

Round $T + 1$. Each player j broadcasts 1 if at all rounds t , all its incoming messages from player $1 \leq i \leq k'$ were disjoint with all its incoming messages from player $i + k'$.

Clearly, after T rounds, player j receives $(x_i)_{(j-1)\lceil n/k \rceil + 1}, \dots, (x_i)_{j\lceil n/k \rceil}$ from every i . Hence, $\text{PWDISJ}(x) = 0$ if and only if a 0 is broadcasted by some player in the last round. Therefore, every player will know the answer after the last round. \square

Theorem 2. $\mathbf{BRound}_b(\text{PWDISJ}) = \Omega(n/b)$.

Proof. It is well-known that, in the two party case $k = 2$, the round complexity of set-disjointness with error probability ϵ is $\Omega(n/b)$ [22]. If $k > 2$ we get the same bound for (P)WDISJ by considering the instance where $x_1 = x \in \{0, 1\}^n$ is given to player 1, $x_{1+k'} = y \in \{0, 1\}^n$ is given to player 2, and the empty set ϕ , represented by $(0, \dots, 0)^T$, is given to all the other $k - 2$ players. \square

Corollary 1. *Let $k = n$. Then, $\mathbf{URound}_1(\text{P)WDISJ}) = 2$ and $\mathbf{BRound}_b(\text{P)WDISJ}) = \Omega(n/b)$.*

Corollary 2. $\Gamma_{(k-1) \times b}^{1 \times b}(\text{P)WDISJ}) = \Omega(k)$.

3 A hierarchy of models according to the range

In previous section we proved that the broadcast ($r = 1$) and the unicast ($r = k - 1$) models are fundamentally different in their power to solve one particular problem. These two models are the two ends of the spectrum of values of the range r . In this section we prove that the sensitivity to the range is more general. In particular, we show that the round complexity of (P)WDISJ is completely sensitive to the range.

Lemma 1. $\text{ROUND}_{r \times b}(\text{P)WDISJ}) = \Omega\left(\frac{n}{\min(kb, rb + \lceil \log r \rceil k)}\right)$.

Proof. We use a reduction from the two-party communication problem $\text{DISJ}_{k'n}$, where instances are pairs (x, y) of boolean vectors, each of length $k'n$. The communication complexity (bits to be exchanged) of $\text{DISJ}_{k'n}$ is $\Theta(k'n)$ [22]. We transform an instance of $\text{DISJ}_{k'n}$ into an instance of (P)WDISJ in the direct way. From (x, y) we define the input (x_1, \dots, x_k) of function (P)WDISJ as follows: $x = x_1 \cdots x_{k'}$ and $y = x_{k'+1} \cdots x_k$. Obviously, $\text{DISJ}_{k'n}(x, y) = 1 \iff \text{P)WDISJ}(x_1, \dots, x_k) = 1$.

Let us consider any protocol \mathbf{P} that solves (P)WDISJ in $T_{\mathbf{P}}$ rounds. If we group players 1 to k' into a global player A and players $k' + 1$ to k into a global player B , protocol \mathbf{P} would yield a protocol for solving $\text{DISJ}_{k'n}$. So the question is the following: How many bits are exchanged between A and B ? Let us derive an upper bound for this.

Consider a player i in A . Player i sends one message of length b to each player in B , thus he sends $k'b$ bits. However, since $r \leq 2^b$, the messages sent by player i to players in B can be compressed as follows. Since player i can send up to r different messages, one can consider that she sends to each player $j \in B$ a message numbered from the set $\{0, 1, \dots, r - 1\}$ that identifies the message $m(i, j)$ sent to player j . These numbers, of $\lceil \log r \rceil$ bits each, can be used to obtain the actual message from a table that contains the r messages, of b bits each, sent by i . Hence, the total number of bits sent by i to B is upper bounded by the length of the k' numbers, $\lceil \log r \rceil k'$ bits, and the size of the message table, br bits; a total of $rb + \lceil \log r \rceil k'$ bits.

Let us define $\beta = \min(kb', rb + \lceil \log r \rceil k')$. In each round, the number of bits exchanged between A and B is upper bounded by $k\beta$. Therefore, considering

that the communication complexity of $\text{DISJ}_{k'n}$ is $\Theta(k'n)$, it follows that $T_{\mathbb{P}}k\beta = \Omega(k'n)$. Therefore, $\text{ROUND}_{r \times b}(\text{PWDISJ}) = \Omega\left(\frac{n}{2\beta}\right)$, as claimed. \square

Lemma 2. $\text{ROUND}_{r \times b}(\text{PWDISJ}) \leq \left\lceil \frac{n}{k \lceil \log r \rceil} \right\rceil + 1$.

Proof. Consider the same protocol used in the proof of Theorem 1 but with messages of $\lceil \log r \rceil \leq b$ bits. \square

Putting these together, we get the following theorem.

Theorem 3. For k sufficiently large and for $r \geq 2$ such that $\frac{rb}{\log r} = O(k)$,

$$\text{ROUND}_{r \times b}(\text{PWDISJ}) = \Theta\left(\frac{n}{k \log r}\right).$$

Proof. The upper bound follows from the previous lemma. For the lower bound, it follows from Lemma 1 that

$$\text{ROUND}_{r \times b}(\text{PWDISJ}) \geq \frac{n}{k \min(b, \lceil \log r \rceil (1 + \frac{2rb}{\lceil \log r \rceil k})} \geq \frac{n}{k \lceil \log r \rceil (1 + \frac{2rb}{\lceil \log r \rceil k})},$$

where the last inequality follows from $\lceil \log r \rceil \leq b$ and $\frac{2rb}{\lceil \log r \rceil k} > 0$. Since $\frac{rb}{\log r} = O(k)$, we deduce that, for k sufficiently large, there is a constant $\Delta > 0$ such that $\frac{2rb}{\lceil \log r \rceil k} \leq \Delta$, and hence

$$\text{ROUND}_{r \times b}(\text{PWDISJ}) \geq \frac{n}{k \lceil \log r \rceil (1 + \Delta)} = \Omega\left(\frac{n}{k \lceil \log r \rceil}\right).$$

\square

The natural way to interpret Theorem 3 is to parametrize everything by k . Following the spirit of the $\mathcal{CONGEST}$ model [29], we are going to restrict both the bandwidth and the range by taking $b = \log k$ and varying r from 2 to $k-1$. Observe that, when $b = \log k$ and $r \leq k-1$, it always holds that $\frac{rb}{\log r} = O(k)$. Hence, the next corollaries are direct consequences of Theorem 3.

Corollary 3. For every n and every constant integer $c \geq 2$, we have

$$\text{ROUND}_{\log k \times \log k}(\text{PWDISJ}) = \Theta\left(\frac{n}{k \log \log k}\right) \quad \text{and} \quad \text{ROUND}_{c \times \log k}(\text{PWDISJ}) = \Theta\left(\frac{n}{k}\right)$$

$$\text{In other words, } \Gamma_{\log k \times \log k}^{c \times \log k}(\text{PWDISJ}) = \Theta(\log \log k).$$

In general, we can state the following corollary.

Corollary 4. For every n and every $r' \geq r \geq 2$, we have

$$\text{ROUND}_{r' \times \log k}(\text{PWDISJ}) = \Theta\left(\frac{n}{k \log r'}\right) \quad \text{and} \quad \text{ROUND}_{r \times \log k}(\text{PWDISJ}) = \Theta\left(\frac{n}{k \log r}\right)$$

$$\text{In other words, } \Gamma_{r' \times \log k}^{r \times \log k}(\text{PWDISJ}) = \Theta\left(\frac{\log r'}{\log r}\right).$$

4 Most functions have maximal round complexity

From the results presented in the previous sections one may be tempted to conclude that, in general, increasing the range r increases the power of the protocols. In particular, one may conclude that the unicast congested clique model has much more power than the broadcast congested clique model (even if in the first we restrict the bandwidth to $b = 1$ while in the latter we allow it to be $b = o(n)$). We show here that this fact, which holds for function PWDISJ, holds for very few other functions. More precisely, we are going to prove that for almost every boolean function f , the broadcasting protocol is optimal. We start by considering deterministic decision protocols that compute functions f correctly (i.e., they make no mistake). (Some proofs are omitted).

Lemma 3. *The number of T -round deterministic decision protocols in the unicast congested clique model CLIQUE-UCAST_b is at most $2^{N(T)}$, where*

$$N(T) = 2^{T(k-1)b+n} \left(1 + \frac{(k+1)(k-1)b}{2^{(k-1)b}}\right).$$

Now, we still consider deterministic protocols, but now we allow them to make mistakes. We say that a deterministic protocol P computes f with error $\epsilon \geq 0$ if it outputs $f(x)$ for at least $(1 - \epsilon)2^{nk}$ of the inputs x of f .

Lemma 4. *Let P be a deterministic decision protocol and let $P(x)$ denote the output of P with input $x \in \{0, 1\}^{nk}$. Let $M_\epsilon(P)$ be the number of functions f which are computed by P with an error $\epsilon > 0$. We have,*

$$M_\epsilon(P) \leq \left(\frac{2e}{\epsilon}\right)^{\epsilon 2^{nk}} = 2^{\log\left(\frac{2e}{\epsilon}\right)\epsilon 2^{nk}}.$$

We show now that a deterministic protocol P that computes a function f chosen uniformly at random with error ϵ requires the maximal number of rounds $\lceil n/b \rceil$ with high probability. Let us extend our notation, so that $\mathbf{UROUND}_b^\epsilon(f)$ is the round complexity of function f when protocols are deterministic and error ϵ is allowed.

Theorem 4. *For k sufficiently large and for every n , and $\epsilon > 0$ such that $1 - \log\left(\frac{2e}{\epsilon}\right)\epsilon > 0$, we have*

$$\Pr\{\mathbf{UROUND}_b^\epsilon(f) = \lceil n/b \rceil\} \geq 1 - 2^{-2^{kn} \left(\frac{1 - \log\left(\frac{2e}{\epsilon}\right)\epsilon}{2}\right)}.$$

For $\epsilon = 0$ (i. e. the case without error), we have

$$\Pr\{\mathbf{UROUND}_b^0(f) = \lceil n/b \rceil\} \geq 1 - 2^{-2^{kn} 0.5}.$$

Proof. Since there are $2^{2^{kn}}$ different functions $f : \{0, 1\}^{kn} \rightarrow \{0, 1\}$, we have

$$\Pr\{\mathbf{UROUND}_b^\epsilon(f) \leq T\} \leq \frac{2^{N(T)} \max_P M_\epsilon(P)}{2^{2^{kn}}}.$$

From Lemmas 3 and 4, for $\epsilon > 0$, we have

$$\begin{aligned} \Pr\{\mathbf{UROUND}_b^\epsilon(f) \leq T\} &\leq 2^{2^{T(k-1)b+n} \left(1 + \frac{(k+1)(k-1)b}{2^{(k-1)b}}\right)} 2^{\log(\frac{2\epsilon}{\epsilon})\epsilon 2^{nk}} 2^{-2^{kn}} \\ &\leq 2^{-2^{kn} \left(1 - \log(\frac{2\epsilon}{\epsilon})\epsilon - 2^{T(k-1)b+n-kn} \left(1 + \frac{(k+1)(k-1)b}{2^{(k-1)b}}\right)\right)}. \end{aligned}$$

For k sufficiently large, the quantity $1 + \frac{(k+1)(k-1)b}{2^{(k-1)b}}$ can be upper bounded (by 2 for example). Now let us take $T = \lceil n/b \rceil - 1$. Then, we have $Tb - n \leq -b$ and, thus

$$2^{T(k-1)b+n-kn} = 2^{(k-1)(Tb-n)} \leq 2^{-b(k-1)} \leq 2^{-k}$$

Thus, for k sufficiently large, the term, $2^{T(k-1)b+n-kn} \left(1 + \frac{(k+1)(k-1)b}{2^{(k-1)b}}\right)$ can be upper bounded by any positive value, in particular by $\frac{1 - \log(\frac{2\epsilon}{\epsilon})\epsilon}{2}$. Thus, we get that

$$\Pr\{\mathbf{UROUND}_b^\epsilon(f) \leq \lceil n/b \rceil - 1\} \leq 2^{-2^{kn} \left(\frac{1 - \log(\frac{2\epsilon}{\epsilon})\epsilon}{2}\right)},$$

which is the result since, for any f , one trivially has $\mathbf{UROUND}_b^\epsilon(f) \leq \mathbf{UROUND}_b^0(f) \leq \lceil n/b \rceil$.

For $\epsilon = 0$ we proceed on the same way, after noticing that $\max_P M_0(P) = 1$. \square

Theorem 5. *For k sufficiently large, for every n , and $0 \leq \epsilon \leq 0.2$, there exists a positive constant $c(\epsilon) > 0$ such that $\Pr\{\mathbf{UROUND}_b^\epsilon(f) = \lceil n/b \rceil\} \geq 1 - 2^{-2^{kn} c(\epsilon)}$.*

Recall that $\mathbf{UROUND}_b(f)$ is the round complexity of computing function f with randomized protocols, which may use public coins, with success probability $1 - \epsilon$. From the previous results we can prove that most functions have round complexity $\lceil n/b \rceil$.

Corollary 5. *For k sufficiently large and for every n , and $0 \leq \epsilon \leq 0.2$, there exists a positive constant $c(\epsilon) > 0$ such that,*

$$\Pr\{\mathbf{UROUND}_b(f) = \lceil n/b \rceil\} \geq 1 - 2^{-2^{kn} c(\epsilon)}.$$

Proof. The result follows from Theorem 5, and Theorem 3.20 at [24] using the uniform distribution as the distribution μ of the inputs. \square

The following bound is obvious for any function f .

$$\Gamma_{(k-1) \times b}^{1 \times b}(f) = \frac{\mathbf{BROUND}_b(f)}{\mathbf{UROUND}_b(f)} \geq \min_f \Gamma_{(k-1) \times b}^{1 \times b}(f) \geq 1.$$

Next corollary, which is a direct consequence of Corollary 5, says that previous inequality is in fact an equality for almost every boolean function.

Corollary 6. *For k sufficiently large and for every n , and $0 \leq \epsilon \leq 0.2$, there exists a positive constant $c(\epsilon) > 0$ such that,*

$$\Pr\{\Gamma_{(k-1) \times b}^{1 \times b}(f) = 1\} \geq 1 - 2^{-2^{kn} c(\epsilon)}.$$

5 Comparing models with different combinations of range and bandwidth for arbitrary boolean functions

In this section we explore the relative round complexities of different modes of the congested clique model with various combinations of range and bandwidth $\Gamma_{r' \times b}^{r \times b}(f)$ for arbitrary boolean functions f . The first result shows that for *most* boolean functions f , $\Gamma_{r' \times b}^{r \times b}(f) = \Theta(b'/b)$.

Theorem 6. *For k sufficiently large and for every n , there is a positive constant $c(\epsilon) > 0$ such that*

$$\Pr \left\{ \Gamma_{r' \times b}^{r \times b}(f) = \lceil n/b \rceil / \lceil n/b' \rceil \right\} \geq 1 - 2^{-2^{kn} c(\epsilon) + 1}.$$

Proof. From Corollary 5, a function f *simultaneously* satisfies $\text{ROUND}_{r \times b}(f) = \lceil n/b \rceil$ and $\text{ROUND}_{r' \times b'}(f) = \lceil n/b' \rceil$ with probability at least $1 - 2^{-2^{kn} c(\epsilon) + 1}$. \square

Now, we show that in fact the typical case shown in the previous theorem is not far from the worst case, studied in the following sequence of results.

Theorem 7. *Let r be such that $r \geq r'$ or $r = 2^b$. Then, for every function f , $\Gamma_{r' \times b}^{r \times b}(f) \leq \lceil b'/b \rceil$.*

Proof. Let P' be a T -round protocol in $\text{CLIQUE-RCAST}_{r' \times b'}$. From P' we construct the protocol P in $\text{CLIQUE-RCAST}_{r \times b}$ as follows. Consider the message $m_t(i, j)$ sent by player i to player j in round t of P' . For each $1 \leq \ell \leq \lceil b'/b \rceil$, let $\text{block}_t^\ell(i, j)$ be the ℓ^{th} block of length b of $m_t(i, j)$. The last block is padded with 0s. For each ℓ and i , we have: $|\{\text{block}_t^\ell(i, j), 1 \leq j \leq k \in N\}| \leq \min\{r', 2^b\} \leq r$.

Then, during round number $(t-1)\lceil b'/b \rceil + \ell$ of P , player i sends to player j the b bits of $\text{block}_t^\ell(i, j)$. The inequalities above ensure that P is a well-defined protocol in $\text{CLIQUE-RCAST}_{r \times b}$. Since P knows the bandwidth b' it can discard the padding bits. The total number of rounds executed by P is $T \lceil b'/b \rceil$. \square

Theorem 8. *Let $b \leq b' \leq n$, and k sufficiently large. Then, there exists a function f such that: $\Gamma_{r' \times b}^{r \times b}(f) = \lceil b'/b \rceil$.*

Proof. Let $b' = n$. In this case, every function $f : (\{0, 1\}^n)^k \rightarrow \{0, 1\}$ can be solved in one round in the model $\text{CLIQUE-RCAST}_{r' \times b'}$. On the other hand, from Corollary 5, almost every function $f : (\{0, 1\}^n)^k \rightarrow \{0, 1\}$ satisfies $\text{ROUND}_{r \times b}(f) = \lceil n/b \rceil = \lceil b'/b \rceil$. When $n > b'$, let us define $n' = b'$. From Corollary 5, almost every function $f' : (\{0, 1\}^{n'})^k \rightarrow \{0, 1\}$ satisfies $\text{ROUND}_{r \times b}(f') = \lceil \frac{n'}{b} \rceil$. Let us take one such function f' , and define a new function $f : (\{0, 1\}^n)^k \rightarrow \{0, 1\}$ as follows: $f(x_1, x_2, \dots, x_k) = f'(y_1, y_2, \dots, y_k)$, where each y_i is the vector formed with the n' first bits of x_i . Hence, $\text{ROUND}_{r \times b}(f) = \text{ROUND}_{r \times b}(f') = \lceil n'/b \rceil = \lceil b'/b \rceil$ while $\text{ROUND}_{r' \times b'}(f) = 1$. \square

Remark 1. When $b|b'$ is a multiple of b and $b'|n$, we have $\lceil n/b \rceil / \lceil n/b' \rceil = (n/b)/(n/b') = b'/b = \lceil b'/b \rceil$. When $n = b'$, we also have $\lceil n/b \rceil / \lceil n/b' \rceil = \lceil b'/b \rceil$. Thus, in the previous cases, for $r \geq r'$ of $r = 2^b$, the maximal value $\lceil b'/b \rceil$ for the value of $\Gamma_{r' \times b'}^{r \times b}(f)$ is reached with high probability. On the other hand, in some cases, there exists a small but intriguing gap between the maximal value $\lceil b'/b \rceil$ and the value $\lceil n/b \rceil / \lceil n/b' \rceil$ reached with high probability. For example, take $b = 2$, $b' = 3$. For $n = 4$, we have $\lceil b'/b \rceil = 2$ and $\lceil n/b \rceil / \lceil n/b' \rceil = 1$.

Note that Theorem 7 holds when $r' \leq r$ or $r = 2^b$. Without this hypothesis we only get the following weaker, general bound.

Theorem 9. *Let $r \geq 2$ and $r' \geq 1$. Then, for every function f ,*

$$\Gamma_{r' \times b'}^{r \times b}(f) \leq \min \left(\left\lceil \frac{b'}{\lfloor \log r \rfloor} \right\rceil, \left\lceil \frac{r'}{r-1} \right\rceil \left\lceil \frac{b'}{b} \right\rceil \right).$$

Observe that the two values of the minimum are complementary, since none implies the other.

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