The Length of the Longest Edge in Multi-dimensional Delaunay Graphs
(Extended Abstract)

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The topic of this work is the study of the length of the longest Delaunay edge in multidimensional Euclidean spaces. In particular, the Delaunay graph considered is defined over a set of points distributed uniformly at random in a multidimensional body of unitary volume. The motivation to study such setting comes from the Random Geometric Graph (RGG) model $\mathcal{G}_{n,r}$, where n nodes are distributed uniformly at random in a unit disk or, more generally, according to some specified density function on d-dimensional Euclidean space [9].

It is known [7] that the length of the longest Delaunay edge is strongly influenced by the boundaries of the enclosing body. For instance, if the area enclosing the points is a disk, the longest edge is asymptotically larger than if the area is the surface of a sphere. Therefore, we study the problem for two cases that we call *with boundary* and *without boundary.*

Multidimensional Delaunay tessellations have been studied before with respect to construction algorithmic techniques [6, 8]. Restricted to two dimensions, upper bounding the length of the longest Delaunay edge has attracted interest recently [7] in the context of extensive algorithmic work [1, 3, 4] aimed to reduce the energy consumption of geographically routing messages in Radio Networks. The upper bounds presented in [7] are only asymptotical, restricted to $d = 2$, and for enclosing bodies with boundary (although without boundary is implicit because the distance to the boundary is parametric).

The results presented here include upper and lower bounds for d-dimensional bodies with and without boundaries, that hold for a parametric error probability ε. Lower bounds without boundary and all upper bounds apply for any $d > 1$. Lower bounds with boundary are shown for $d \in \{2, 3\}$. The results shown are asymptotically tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$, for any constant $c > 0$. To the best of our knowledge, this is the first comprehensive study of this problem.

In the following section, some necessary notation is introduced. Upper and lower bounds for enclosing bodies without boundaries are enumerated in Section 2, and the case with boundaries is covered in Section 3. We conclude with some open problems. Upper bounds are proved exploiting that, thanks to the uniform density, is very unlikely that a "large" area/volume is void of points. Lower bounds, on the other hand, are proved showing that some configuration that yields a Delaunay edge of certain length is not very unlikely. Complete proofs can be found in [2].

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1 Preliminaries

The following notation will be used throughout. A *d-sphere*, of radius r is the set of all points in a d-dimensional L_2-space that are located at distance r (called the *radius*) from a given point (called the *center*). A *d-ball*, of radius r is the set of all points in a d-dimensional L_2-space that are located at distance *at most* r (called the *radius*) from a given point (called the *center*). The *area* of a sphere is the area of its surface. The *volume* of a ball is the amount of space it occupies. A *unit sphere* is a sphere of area 1. A *unit ball* is a ball of volume 1.

Let P be a set of points on a d-sphere. Given two points $a, b \in P$, let $\widehat{ab}$ be the arc of a great $(d-1)$-ball intercepted between them. Let $\delta(a, b)$ be the orthodromic distance of such arc. Let the *orthodromic diameter* be the longest orthodromic distance between any pair of points in the surface area of an spherical cap. Let $A_d(x, y)$ be the surface area of a spherical cap of orthodromic diameter y, of a d-sphere of surface area x. Let $V_d(x, y)$ be the volume of a spherical cap of base diameter y, of a d-ball of volume x. Let $D(P)$ be the Delaunay graph of a set of points P.

The following definitions of a Delaunay graph of a set of points in d-dimensional bodies can be derived as in Theorem 9.6.ii in [5].

Definition 1. *Let P be a set of points in a d-sphere, two points $a, b \in P$ form an arc of $D(P)$, if and only if there is a d-dimensional spherical cap C such that, with respect to the surface of the cap, it contains a and b on the boundary and does not contain any other point of P.*

Definition 2. *Let P be a set of points in a d-ball, two points $a, b \in P$ form an edge of $D(P)$, if and only if there is a d-ball B such that, a and b are located in the surface area of B, and the interior of B does not contain any other point of P.*

2 Enclosing Body without Boundary

The following theorems show upper and lower bounds on the length of arcs in the Delaunay graph on a d-sphere.

Theorem 3. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-sphere, with probability at least $1 - \varepsilon$, for $0 < \varepsilon < 1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$A_d(1, \delta(a, b)) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Corollary 4. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit circumference (2-sphere), with probability at least $1 - \varepsilon$, for $0 < \varepsilon < 1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a, b) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Corollary 5. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit sphere (3-sphere), with probability at least $1 - \varepsilon$, for $0 < \varepsilon < 1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a, b) \geq \frac{1}{\sqrt{\pi}} \arccos\left(1 - \frac{2\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}\right).$$

Theorem 6. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-sphere, with probability at least ε, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that $A_d(1, \delta(a,b)) \geq A_d(1, \rho_1)$, where*

$$A_d(1, \rho_1) = \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n - 2 + \ln\left((e-1)/(e^2\varepsilon)\right)},$$

for any $0 < \varepsilon < 1$ such that $A_d(1, 2\rho_1) \leq 1 - 1/(n-1)$.

Corollary 7. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit circumference (2-sphere), with probability at least ε, for any $e^{1-n-4/n} \leq \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n - 2 + \ln\left((e-1)/(e^2\varepsilon)\right)}.$$

Corollary 8. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit sphere (3-sphere), with probability at least ε, for any $e^{-n+2\sqrt{n-1}-1} \leq \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{1}{\sqrt{\pi}} \arccos\left(1 - \frac{2\ln\left((e-1)/(e^2\varepsilon)\right)}{n - 2 + \ln\left((e-1)/(e^2\varepsilon)\right)}\right).$$

3 Enclosing Body with Boundary

The following theorems show upper and lower bounds on the length of edges in the Delaunay graph on a d-ball.

Theorem 9. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-ball, with probability at least $1 - \varepsilon$, for $0 < \varepsilon < 1$, there is no edge $(a, b) \in D(P)$, $a, b \in P$, such that*

$$V_d(1, ||\overrightarrow{a,b}||_2) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Corollary 10. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit disk (2-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}e^{-\sqrt{2}(n-2)/\pi} < \varepsilon < 1$, there is no edge $(a, b) \in D(P)$, $a, b \in P$, such that*

$$||\overrightarrow{a,b}||_2 \geq \sqrt[3]{\frac{16}{\sqrt{\pi}} \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}}.$$

Corollary 11. *Given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit ball (3-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}e^{-2(n-2)\big/(3\sqrt{\pi})} < \varepsilon < 1$, there is no edge $(a, b) \in D(P)$, $a, b \in P$, such that*

$$||\overrightarrow{a,b}||_2 \geq \sqrt[4]{\frac{96}{\pi^{3/2}} \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}}.$$

Theorem 12. *For $d = 2$, given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that $||\overrightarrow{ab}||_2 \geq \rho_1/2$, where*

$$V_d(1, \rho_1) = \frac{\ln(\alpha/\varepsilon)}{(n - 2 + \ln(\alpha/\varepsilon))},$$

where $\alpha = (1 - e^{-1/16})(1 - e^{-1/32})e^{-1}$, for any $0 < \varepsilon \leq \alpha/e^2$ such that $V_d(1, \rho_1) \leq 1/2 - 1/n$.

Theorem 13. *For $d = 3$, given the Delaunay graph $D(P)$ of a set P of $n > 4$ points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that $||\overrightarrow{ab}||_2 \geq \rho_1/2$, where*

$$V_d(1, \rho_1) = \frac{\ln(\alpha/\varepsilon)}{(n - 2 + \ln(\alpha/\varepsilon))},$$

where $\alpha = (1 - e^{-1/6})(1 - e^{-1/12})e^{-12}$, for any $0 < \varepsilon \leq \alpha/e$ such that $V_d(1, \rho_1) \leq 1/2 - 1/n$.

4 Open Problems

It would be interesting to extend this study to other norms, such as L_1 or L_∞. Also, Theorems 12 and 13 were proved showing that some configuration that yields a Delaunay edge of some length is not unlikely. Different configurations were used for each, but a configuration that works for both cases exists (although yielding worse constants). We conjecture that (modulo some constant) the same bound can be obtained in general for any $d > 1$. Both questions are left for future work.

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